Conics through Inflections of Self-projective Quartics.

By F. R. Sharpe.

The line through any 2 of the 9 inflections of a plane cubic curve meets the curve again in another inflection. In 1875 J. Grassmann * stated the analogous theorem for quartic curves: "The conic through any 5 of the 24 inflections of a quartic curve meets the curve again in 3 other inflections. There are therefore 759 such conics." This theorem is now known to be untrue. In 1899 Ciani,† in a paper on quartics invariant under homologies, found quartics having as many as 51 conics each passing through 8 inflections. In 1901 the same author ‡ showed that the Klein quartic had 147 such conics as well as 112 conics through 6 inflections. His method depended chiefly on the use of groups of collineations under which the quartics and conics were invariant. In this paper more use is made of the points of inflection themselves. The results confirm those of Ciani as regards the numbers of conics through 8 inflections. They also show the existence of additional conics through 6 inflections, 220 in the case of a quartic invariant under 1 homology and 2100 for the Klein quartic.

The different cases were classified by Ciani according to the number of homologies under which the quartic is invariant. For the sake of clearness let us begin with the quartic

$$x^4 + y^4 + z^4 - 3\mu\,(y^2 z^2 + z^2 x^2 + x^2 y^2) = 0, \tag{10}$$

which is invariant under a G_{24} of collineations 9 of which are homologies. If (f, g, h) are the coordinates of one point of inflection, the 24 inflections are §

$$\begin{array}{lll}
A \equiv (f, g, h), & B \equiv (g, h, f), & C \equiv (h, f, g), \\
D \equiv (-f, g, h), & K \equiv (-g, h, f), & R \equiv (-h, f, g), \\
S \equiv (f, -g, h), & E \equiv (g, -h, f), & L \equiv (h, -f, g), \\
N \equiv (f, g, -h), & U \equiv (g, h, -f), & G \equiv (h, f, -g), \\
X \equiv (f, h, g), & J \equiv (g, f, h), & Q \equiv (h, g, f), \\
O \equiv (-f, h, g), & V \equiv (-g, f, h), & H \equiv (-h, g, f), \\
M \equiv (f, -h, g), & T \equiv (g, -f, h), & F \equiv (h, -g, f), \\
P \equiv (f, h, -g), & W \equiv (g, f, -h), & I \equiv (h, g, -f).
\end{array}$$

* Dissertation, Berlin. † *Rend. Circ. Mat. di Palermo.* ‡ *Annali di Mat.*

§ Ciani, *Annali di Mat.*, 1901, p. 53.

The reason for the particular choice of letters will appear later on in the work.

Corresponding to the 9 homologies there are 9 systems of conics:

$$p\,x^2 + q\,y^2 + r\,z^2 + s\,y\,z = 0, \tag{1}$$
$$p\,y^2 + q\,z^2 + r\,x^2 + s\,z\,x = 0, \tag{2}$$
$$p\,z^2 + q\,x^2 + r\,y^2 + s\,x\,y = 0, \tag{3}$$
$$p\,(y^2 + z^2) + q\,x^2 + r\,y\,z + s\,x\,(y + z) = 0, \tag{4}$$
$$p\,(z^2 + x^2) + q\,y^2 + r\,z\,x + s\,y\,(z + x) = 0, \tag{5}$$
$$p\,(x^2 + y^2) + q\,z^2 + r\,x\,y + s\,z\,(x + y) = 0, \tag{6}$$
$$p\,(y^2 + z^2) + q\,x^2 + r\,y\,z + s\,x\,(y - z) = 0, \tag{7}$$
$$p\,(z^2 + x^2) + q\,y^2 + r\,z\,x + s\,y\,(z - x) = 0, \tag{8}$$
$$p\,(x^2 + y^2) + q\,z^2 + r\,x\,y + s\,z\,(x - y) = 0, \tag{9}$$

each being invariant under 1 homology. The inflections may be arranged in the corresponding invariant pairs of points:

	AD	*SN*	*XO*	*MP*		*AS*	*DN*	*XM*	*OP*		*AN*	*SD*	*XP*	*OM*
(1′)	*BK*	*EU*	*JV*	*TW*	(2′)	*BE*	*KU*	*JT*	*VW*	(3′)	*BU*	*EK*	*JW*	*VT*
	CR	*LG*	*QH*	*FI*		*CL*	*RG*	*QF*	*HI*		*CG*	*LR*	*QI*	*HF*
	AX	*DO*	*SP*	*NM*		*AQ*	*DI*	*SF*	*NH*		*AJ*	*DT*	*SV*	*NW*
(4′)	*BJ*	*KV*	*EW*	*UT*	(5′)	*BX*	*KP*	*EM*	*UO*	(6′)	*BQ*	*KF*	*EH*	*UI*
	CQ	*RH*	*LI*	*GF*		*CJ*	*RW*	*LT*	*GV*		*CX*	*RM*	*LO*	*GP*
	AO	*DX*	*SM*	*NP*		*AF*	*DH*	*SQ*	*NI*		*AW*	*DV*	*ST*	*NJ*
(7′)	*BV*	*KJ*	*ET*	*UW*	(8′)	*BM*	*KO*	*EX*	*UP*	(9′)	*BI*	*KH*	*EF*	*UQ*
	CH	*RQ*	*LF*	*GI*		*CT*	*RV*	*LJ*	*GW*		*CP*	*RO*	*LM*	*GX*

If from any 1 of these 9 sets of pairs of points we select any 3 pairs, the 6 points lie on a conic of the corresponding type. We can now take up with advantage the various types of quartics, beginning with

CASE I. One homology, $\begin{pmatrix} x & y & z \\ -x & y & z \end{pmatrix}$, the quartic being

$$x^4 + x^2 f_2(y, z) + f_4(y, z) = 0.$$

Using the arrangement (1′) there are 220 conics of type (1) that pass through 6 inflections.

CASE II. Three homologies, $\begin{pmatrix} x & y & z \\ x & z & y \end{pmatrix}$, $\begin{pmatrix} x & y & z \\ z & y & x \end{pmatrix}$, $\begin{pmatrix} x & y & z \\ y & x & z \end{pmatrix}$, with their centers on $x + y + z = 0$ and their axes concurrent at $(1, 1, 1)$. The quartic is of the symmetric form

$$a\,\Sigma\,x^4 + b\,\Sigma\,y^2 z^2 + c\,x\,y\,z\,\Sigma\,x + d\,\Sigma\,x\,y \cdot \Sigma\,x^2 = 0.$$

Using the arrangements (4′), (5′), (6′) and the corresponding types of conics,

the 4 sets of 6 points *ABCJQX*, *DOLITU*, *FGPKVS*, *EHMNRW* are common to the 3 arrangements and lie on 4 conics of the symmetric type $a\Sigma x^2 + b\Sigma xy = 0$. The line $x + y + z = 0$ is a bitangent to the quartic, the points of contact being $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$, where $\omega^3 = 1$, which also lie on the 4 conics. Hence, the inflections lie on 3 sets of 220 conics, 4 of which are common to the 3 sets and pass through the points of contact of a bitangent.

CASE III. Three homologies, $\begin{pmatrix} x & y & z \\ -x & y & z \end{pmatrix}$, $\begin{pmatrix} x & y & z \\ x & -y & z \end{pmatrix}$, $\begin{pmatrix} x & y & z \\ x & y & -z \end{pmatrix}$, the quartic being

$$a x^4 + b y^4 + c z^4 + d y^2 z^2 + e z^2 x^2 + f x^2 y^2 = 0.$$

The inflections may be arranged in 6 sets of 4,

ADNS, *BKEU*, *CRLG*, *XOMP*, *JVTW*, *QHFI*,

such that a conic of the form

$$p x^2 + q y^2 + r z^2 = 0 \tag{10}$$

passes through each pair of sets. Hence, the inflections lie in eights on 15 conics. Among the conics of types (1), (2), (3) each of these conics occurs 4 times. Hence, the inflections lie in sixes on 3 sets of 160 conics.

CASE IV. Five homologies, the quartic being

$$a (x^4 + y^4) + b z^4 + c z^2 (x^2 + y^2) + d x^2 y^2 = 0.$$

In addition to the 3 homologies of Case III, we have also $\begin{pmatrix} x & y & z \\ y & x & z \end{pmatrix}$, $\begin{pmatrix} x & y & z \\ y & x & -z \end{pmatrix}$. The inflections may be arranged in 6 new sets of 4,

AJNW, *BIQU*, *CGPX*, *DSTV*, *EFHK*, *LMOR*,

such that a conic of the form

$$p (x^2 + y^2) + q z^2 + r x y = 0 \tag{11}$$

passes through each pair of sets. Three of these pairs coincide with 3 of the pairs of Case III. Hence, the inflections lie in eights on 27 conics. In counting the conics through 6 inflections of type (1) or (2), each of the 15 conics of type (10) through 8 inflections is counted 4 times, and similarly for (6) or (9) or (11). In the case of type (3) each of the 27 conics of type (10) or (11) is counted 4 times. Hence, the inflections lie in sixes on 4 sets of 160 conics and 1 set of 112 conics.

CASE V. Seven homologies, the quartic being

$$a (x^4 + y^4) + b z^4 + c x^2 y^2 = 0.$$

The 2 additional homologies are

$$\begin{pmatrix} x & y & z \\ y & -x & iz \end{pmatrix}, \begin{pmatrix} x & y & z \\ -y & x & iz \end{pmatrix}.$$

The 24 inflections consist of 4 sets of 4,

$$ANBU,\ DSKE,\ JWQI,\ VTHF,$$

lying on 4 lines through $(0, 0, 1)$, and 4 undulation points on $z = 0$,

$$\begin{cases} C \\ G \end{cases} \begin{cases} L \\ R \end{cases} \begin{cases} O \\ M \end{cases} \begin{cases} X \\ P \end{cases}$$

each of which counts for 2 inflections. The first and fourth lines, the second and third, and each of the 4 taken with $z = 0$ twice, form 10 new degenerate conics through 8 inflections. Hence, the inflections lie in eights on 37 conics. In counting the conics through 6 inflections we find, of type (1), (2), (6) or (9), 32 not passing through undulations, 48 through 2 undulations, 16 through 4 undulations, the 2 latter types being counted twice. Among the conics of type (3) are 8 not through undulations, 80 tangent to the quartic at 1 undulation and 16 tangent at 2 undulations. There are also, corresponding to the 2 new homologies, the 2 new types of conics

$$p z^2 + q (x^2 - y^2) + r x y + s z (x - i y) = 0,$$
$$p z^2 + q (x^2 - y^2) + r x y + s z (y - i x) = 0,$$

and the corresponding pairs of invariant points

$$\begin{matrix} AF & HN \\ BV & IS \end{matrix} \begin{cases} C \\ G \end{cases} \begin{cases} M \\ O \end{cases} \qquad \begin{matrix} AH & EW \\ BT & JK \end{matrix} \begin{cases} C \\ G \end{cases} \begin{cases} M \\ O \end{cases}$$
$$\begin{matrix} DQ & KW \\ EJ & TU \end{matrix} \begin{cases} X \\ P \end{cases} \begin{cases} R \\ L \end{cases} \qquad \begin{matrix} DI & QS \\ FN & UV \end{matrix} \begin{cases} X \\ P \end{cases} \begin{cases} R \\ L \end{cases}$$

In either case, if we select 3 out of the first 8 pairs and reject the 2 conics through 8 points *ANBUVTHF*, *DSKEJWQI*, we find 48 conics. If we select 2 of the first 8 pairs and 2 of the undulations *CM* or *XR*, each counted twice, we find 56 repeated conics. If we select 1 of the first 8 pairs and 4 undulations, we find 8 conics repeated 4 times or 16 conics tangent at 2 undulations. Hence, the inflections lie in sixes on 4 sets of 160 conics, 1 set of 104 conics and 2 sets of 152 conics.

CASE VI. Nine homologies, the quartic being

$$\Sigma x^4 - 3 \mu \Sigma y^2 z^2 = 0.$$

This is a combination of II and III, and differs from IV in having 2 new sets of conics through 8 inflections:

$$p (y^2 + z^2) + q x^2 + r y z = 0, \tag{12}$$

with the 6 sets of 4 points

$$ADXO,\ BKJV,\ CRQH,\ SNMP,\ EUTW,\ LGFI,$$

and

$$p (z^2 + x^2) + q y^2 + r x z = 0, \tag{13}$$

with the corresponding sets of points

$$ASQF,\ BEXM,\ CLJI,\ POKU,\ VWGR,\ DNIH,$$

each giving 12 new and 3 old conics through 8 inflections. Hence, the inflections lie in eights on 51 conics. The sets of conics through 6 inflections of types (4), (8), (9) have 4 conics in common, of the form

$$p\,(x^2+y^2+z^2)+q\,(-yz+zx+xy)=0,$$

that pass through the points of contact of the bitangent $y+z-x=0$; namely,

$$AXFEWG,\ BJMLNI,\ CQTSPU,\ DOHKVR.$$

Similarly, for (5), (7), (9) and the bitangent $x+z-y=0$ we find

$$AQORUW,\ BXVDGI,\ CJHKPN,\ SFMLTE;$$

and for (6), (7), (8) and the bitangent $x+y-z=0$,

$$AJOKLF,\ BQVRMS,\ CXHDTE,\ NWUIGP.$$

Hence, the inflections lie in sixes on 3 sets of 112 conics, on 6 sets of 152 conics and on 4 sets of 4 conics that pass through the points of contact of a bitangent.

CASE VII. Twenty-one homologies, the quartic being

$$\Sigma x^4 - 3\mu \Sigma y^2 z^2 = 0, \tag{13}$$

where $\mu^2-\mu+2=0$.

The quartic of VI is invariant under a G_{24} of collineations, while (13), for the special values of μ, is invariant under a G_{168} of collineations. To determine the systems of conics, we must know how one of the 48 collineations of period 7 interchanges the 24 inflections. The Klein quartic

$$x^3 z + y^3 x + z^3 y = 0 \tag{14}$$

has $A\,(1,0,0)$, $B\,(0,1,0)$, $C\,(0,0,1)$ for 3 inflections. It is invariant under 21 homologies, 1 of which is, in the notation of Weber ("Algebra," Band II),

$$\left.\begin{aligned} x' &= \alpha x + \beta y + \gamma z, \\ y' &= \beta x + \gamma y + \alpha z, \\ z' &= \gamma x + \alpha y + \beta z, \end{aligned}\right\} \tag{15}$$

where

$$\alpha = h\,(\varepsilon^4-\varepsilon^3), \quad \beta = h\,(\varepsilon^2-\varepsilon^5), \quad \gamma = h\,(\varepsilon-\varepsilon^6),$$

$$h = -\frac{1}{\sqrt{-7}}, \quad \varepsilon^7=1, \quad \alpha\varepsilon^2+\beta\varepsilon+\gamma\varepsilon^4=0, \quad \alpha\varepsilon^5+\beta\varepsilon^6+\gamma\varepsilon^3=0.$$

The center of this homology is $a\,(\beta\gamma,\ \alpha\beta,\ \alpha\gamma)$, and the axis bc is

$$\beta\gamma x + \alpha\beta y + \alpha\gamma z = 0. \tag{16}$$

Transforming by the collineations

$$\begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix}, \tag{17}$$

$$\begin{pmatrix} x & y & z \\ \varepsilon x & \varepsilon^2 y & \varepsilon^4 z \end{pmatrix}, \tag{18}$$

we can find 20 other axes and centers of homology. Four centers lie on each axis, 4 axes pass through each center. Transforming A, B, C by means of (15) and (18), we can find the remaining 21 inflections. The quartic (14), when referred to $a\,b\,c$ as triangle of reference (where b, c are 2 of the 4 centers on (16) such that each lies on the axis corresponding to the other), will have (13) for its equation, provided that the other 2 centers, d, d' on $b\,c$ and e, e' on $a\,c$, are transformed into $(0, 1, \pm 1)$, $(1, 0, \pm 1)$. We can then determine the effect of (18) on the inflections when referred to the triangle of reference $a\,b\,c$.

If

$$b \text{ is } (\alpha\beta\varepsilon,\ \alpha\gamma\varepsilon^2,\ \beta\gamma\varepsilon^4),$$
$$a\,c \text{ is } \alpha\beta\varepsilon^6 x + \alpha\gamma\varepsilon^5 y + \beta\gamma\varepsilon^3 z = 0,$$

then

$$c \text{ is } (\alpha\gamma\varepsilon^3,\ \beta\gamma\varepsilon^6,\ \alpha\beta\varepsilon^5),$$

and

$$a\,b \text{ is } \alpha\gamma\varepsilon^4 x + \beta\gamma\varepsilon y + \alpha\beta\varepsilon^2 z = 0.$$

These results follow from the identities (15) Hence, the transformation which sends (14) into (13) is

$$x' = p\,(\beta\gamma x + \alpha\beta y + \alpha\gamma z),$$
$$y' = q\,(\alpha\beta\varepsilon^6 x + \alpha\gamma\varepsilon^5 y + \beta\gamma\varepsilon^3 z),$$
$$z' = r\,(\alpha\gamma\varepsilon^4 x + \beta\gamma\varepsilon y + \alpha\beta\varepsilon^2 z).$$

The 2 remaining centers on $b\,c$ are

$$d\,(\alpha\gamma\varepsilon^4,\ \beta\gamma\varepsilon,\ \alpha\beta\varepsilon^2) \quad\text{and}\quad d'\,(\alpha\beta\varepsilon^6,\ \alpha\gamma\varepsilon^5,\ \beta\gamma\varepsilon^3),$$

and on $a\,c$

$$e\,(\beta\gamma\varepsilon^2,\ \alpha\beta\varepsilon^4,\ \alpha\gamma\varepsilon) \quad\text{and}\quad e'\,(\alpha\gamma\varepsilon^6,\ \beta\gamma\varepsilon^5,\ \alpha\beta\varepsilon^3).$$

Using the additional relations

$$\left.\begin{array}{c} -\mu = \varepsilon^6 + \varepsilon^5 + \varepsilon^3, \quad -\mu' = \varepsilon + \varepsilon^2 + \varepsilon^4, \\ \mu + \mu' = 1, \quad \mu\mu' = 2, \quad h\,(\mu - \mu') = 1, \quad \gamma = \alpha\beta - \gamma^2, \\ \alpha\varepsilon + \beta\varepsilon^4 + \gamma\varepsilon^2 = 1, \quad \alpha = \beta\gamma - \alpha^2, \\ \alpha\varepsilon^6 + \beta\varepsilon^3 + \gamma\varepsilon^5 = 1, \quad \beta = \alpha\gamma - \beta^2, \end{array}\right\} \tag{19}$$

it can be shown that d is transformed into

$$(0,\quad q\,h\,(3+\mu),\quad r\,h\,(3+\mu)).$$

Hence $q = r$, and similarly, by considering e, we find that $p = r$. The required transformation is therefore

$$\left.\begin{aligned} x' &= \beta\gamma x + \alpha\beta y + \alpha\gamma z, \\ y' &= \alpha\beta\varepsilon^6 x + \alpha\gamma\varepsilon^5 y + \beta\gamma\varepsilon^3 z, \\ z' &= \alpha\gamma\varepsilon^4 x + \beta\gamma\varepsilon y + \alpha\beta\varepsilon^2 z. \end{aligned}\right\} \tag{20}$$

Denote $\beta\gamma$, $\alpha\beta\varepsilon^6$, $\alpha\gamma\varepsilon^4$ by f, g, h. The collineation (15) sends

$$A\ (1, 0, 0), \quad B\ (0, 1, 0), \quad C\ (0, 0, 1)$$

into

$$D\ (\alpha, \beta, \gamma), \quad E\ (\beta, \gamma, \alpha), \quad C\ (\gamma, \alpha, \beta).$$

Repeated applications (18) send

$$\begin{array}{lll} D & \text{into} & EFGHIJD, \\ K & \text{into} & LMNOPQK, \\ R & \text{into} & STUVWXR. \end{array}$$

Referred to the new coordinate system, we find the coordinates to be as in Case VI. The reason for the choice of letters is now apparent.

The transformation (18) sends $a\,(\beta\gamma, \alpha\beta, \alpha\gamma)$ into $(\beta\gamma\varepsilon, \alpha\beta\varepsilon^2, \alpha\gamma\varepsilon^4)$. Referred to the new coordinate system, it sends $a\,(1, 0, 0)$ into $(\mu, -1, 1)$, and similarly,

$$\begin{array}{llll} b\ (0, 1, 0) \ \text{into} \ (\mu, 1, -1), & & c\ (0, 0, 1) \ \text{into} \ (0, 1, 1), \\ d\ (0, 1, 1) \ \text{into} \ (-1, \mu, 1), & & e\ (1, 0, 1) \ \text{into} \ (1, \mu, 1). \end{array}$$

These results follow from (20) by using the identities (15) and (19). Hence, proceeding as in the determination of (20), we find the transformation of period 7 to be

$$\left.\begin{aligned} x &= -2x' + (y' - z')\,\mu, \\ y &= 2x' + (y' - z')\,\mu, \\ z &= \mu^2\,(y' + z'). \end{aligned}\right\} \tag{21}$$

Instead of transforming the conics directly by (21), it is more convenient to use the points of inflection themselves. Through A pass 3 conics

$$ADNSXOMP, \quad ADNSJTVW, \quad ADNSQHFI$$

of type (10). If we transform these by (21) and $\begin{pmatrix} x\,y\,z \\ y\,z\,x \end{pmatrix}$, we find 63 conics that are repeated in the types (11), (12), (13) and their transforms. The 24 inflections may be arranged as the vertices of 8 inflectional triangles

$$ABC, \ DKR, \ SEL, \ TFM, \ UGN, \ VHO, \ WIP, \ XJQ.$$

Hence, the inflections lie in eights on 63 conics that pass through one vertex of each triangle.

Through A and B pass 2 conics $ADNSBEKU$, $AJNWBIQU$ of types (10) and (11). If we transform these by (21) and $\begin{pmatrix} x\,y\,z \\ y\,z\,x \end{pmatrix}$, we have 42 conics.

If we transform these by $\begin{pmatrix} x & y & z \\ x & z & y \end{pmatrix}$, we have in all 84 conics that are repeated in (10), (11), (12), (13). Hence, the inflections lie in eights on 84 conics that pass through 2 vertices of 4 triangles.

If we transform (4′), (5′), (6′), (7′), (8′) and (9′) by repeated application of (21), (6′) and (8′) are transformed into (1′), (4′) and (9′) into (2′), (5′) and (7′) into (3′).

The 4 conics common to (4′), (5′), (6′) are transformed into 28 conics common to (1′), (2′), (3′) that pass through the 6 vertices of 2 triangles and the points of contact of a bitangent. The 4 conics common to (4′), (8′), (9′) are similarly transformed into 28 conics common to (1′), (2′). Similarly for (5′), (7′), (9′) and (2′), (3′), also (6′), (7′), (8′) and (1′), (3′). In each case the conics pass through 1 vertex of 6 triangles and the points of contact of a bitangent.

This accounts for the transformations of 12 of the 112 conics through 6 inflections of types (1′), (2′), (3′). The remaining conics can be divided into 2 types. The first type pass through 2 vertices of 2 triangles and 1 vertex of 2 other triangles. Type (1′) has 60 of these conics. If we transform these by (21) and $\begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix}$, we find in all 1260 of these conics. The second type pass through 1 vertex of 6 triangles. Type (1′) has 40 of these conics, excluding the 8 already considered. Transforming these by (21) and $\begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix}$, we find 840 conics. Hence, the inflections lie in sixes on 4 sets of 28 conics, on 1 set of 1260 conics and on 1 set of 840 conics.

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